ENTRAINMENT OF AIR INTO A CUMULUS CLOUD

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ABSTRACT

A theory of convective clouds is presented, the fundamental hypothesis being that the ascending current in a cloud entrains air from its surroundings. A method is developed for computing the amount of entrainment from a knowledge of the temperature and specific humidity inside and outside the cloud. The concentration of water in the form of drops is also determined. Finally the theory is applied to some observations of trade cumulus made near San Juan, Puerto Rico.

1. Introduction

The picture of a cloud usually given [3, pp. 125-144] is that of a column or parcel of moist air ascending wet-adiabatically with no lateral mixing through an environment at rest. A more realistic picture is that given by Bjerknes [1]—the so-called 'slice method,' in which the environment is allowed to subside in order to preserve mass continuity.

The principle of the present theory is illustrated in fig. 1. As the cloud rises, air from the environment is drawn in and mixed with the cloud substance, thus modifying the mechanical and thermodynamic characteristics of the cloud. A steady state is assumed. Two important effects of the entrainment are the following:

(a) The lapse rate within the cloud is not wet-adiabatic as previous theories predict.
(b) The vertical gradient of the concentration of liquid water within the cloud is different from that derived on the basis of other theories.

In some ways this notion of cumulus-cloud convection is similar to the jet-stream theories found so useful in aerodynamics by Tollmien [2, pp. 592-600] and others and in oceanography by Rossby [4]. The basic idea of the jet stream is that whenever a jet or current of fluid moves through a large body of fluid at rest there is entrainment of surrounding fluid into the moving portion, so that the mass flux increases with distance downstream. The theory of jets in simple liquids has not been extended to such relatively complicated conditions as those obtaining in the atmosphere over the range of altitudes and conditions in which cumulus clouds may appear.

The present study entails an important simplification which is not made in the usual jet-stream theories. It is assumed that there is horizontal homogeneity both inside and outside the cloud at each level, the change at the cloud boundary being discontinuous.

In the following paragraph the mass entrainment of outside air into the cloud column is computed as a function of the temperature and specific-humidity soundings inside and outside the cloud.

2. The mass flux gradient in a cloud

Let \( M \) be the upward mass flux across the entire cloud column at the level where the pressure is \( p \). Let the flux of outside air into the cloud between the level \( p \) and a higher level \( p - \delta p \) be \( \delta M \). Since a steady state has been assumed, the flux at \( p - \delta p \) is \( M + \delta M \). The quantity \( M \) is a function of pressure. The quantity \( \delta p \) is an infinitesimal.

Let the actual soundings within the cloud be \( T(p) \) and \( q(p) \), the dry-bulb temperature and the specific humidity respectively—known as functions of pressure. Let the actual soundings in the clear, outside the cloud, be represented by the primed quantities \( T'(p) \) and \( q'(p) \). The fundamental problem is to discover the functional form of \( M(p) \) that makes possible the observed values \( T(p) \) and \( q(p) \) on the supposition that outside air is drawn into the cloud.

The properties within the cloud at some particular level \( p \) are temperature \( T \), specific humidity \( q \), and concentration of liquid water \( W \). At \( p - \delta p \) the mass flux of the cloud is \( M + \delta M \), the temperature is

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$T + \delta T$, the specific humidity is $q + \delta q$, and the liquid-water content is $W + \delta W$.

In rising from $p$ to $p - \delta p$ the cloud undergoes a rather complex transformation which may be replaced by three simple steps. First, the temperature decreases because of the adiabatic decrease of pressure. Second, outside air is entrained and mixed with the cloud at the pressure $p - \delta p$. Third, a phase change achieves equilibrium between vapor and liquid phases.

It is necessary to frame the three steps into differential equations which express the transformation in properties.

In the first step the air rises, dry-adiabatically and with no condensation, from $p$ to $p - \delta p$. The temperature changes from $T$ to $T[(p - \delta p)/p]^{R/c_p}$ where $R$ is the gas constant of moist air and $c_p$ is the specific heat of air at constant pressure. It follows from the binomial theorem and from neglect of infinitesimals of order greater than one that

$$T\left[ \frac{p - \delta p}{p} \right]^{R/c_p} = T\left[ 1 - \frac{R \delta p}{c_p p} \right].$$

The second step is the process of entrainment and mixing. The flux $\delta M$ actually entrained in the pressure interval from $p$ to $p - \delta p$ is assumed to be entrained entirely at $p - \delta p$ and there mixed with the upward flux $M$. The temperature and specific humidity which result from this mixing at constant pressure are designated $T^*$ and $q^*$. They are fictitious quantities. The point $(q^*, T^*)$ is located somewhere along the straight line, called the mixing line, that passes through the two points

$(q' + \delta q', T' + \delta T')$ and $(q, T[1 - (R/c_p)(\delta p/p)])$.

The position of $(q^*, T^*)$ on the mixing line is related to the ratio $\delta M/M$. The mass of water vapor reaching $p - \delta p$ per unit time in the portion of the cloud rising from $p$ to $p - \delta p$ must be $Mq$. The mass of water vapor introduced per unit time by the entrained air must be $(q' + \delta q')\delta M$. The sum of these must, by conservation, be equal to $(M + \delta M)q^*$, the amount in the final uncondensed mixture; hence, if second-order terms are dropped,

$$(M + \delta M)q^* = Mq + q' \delta M. \quad (1)$$

In an isobaric mixing process without phase change the heat is also conserved. The heat per unit time of the unmixed cloud air at $p - \delta p$ is $c_p M T\left[1 - (R/c_p)(\delta p/p)\right]$; that of the entrained portion is $c_p M(T' + \delta T')$. The sum of these must equal the heat of the mixture, $c_p(M + \delta M)T^*$. The second-order terms may be dropped, so that

$$(M + \delta M)T^* = MT - MT(R/c_p)(\delta p/p) + \delta M T'. \quad (2)$$

The third step is the one in which condensation occurs in the mixture. Any condensation process at constant pressure in the $q,T$-plane must, to a first degree of approximation, occur along a straight line of slope $dT/dq = -L/c_p = -2.47 \times 10^8 \degree C$. This line must pass through the final, known point $(q + \delta q, T + \delta T')$ and is therefore determined. It intersects the mixing line. The equation of the condensation line may be written as

$$L(q + \delta q - q^*) = -c_p(T + \delta T' - T^*). \quad (3)$$

The fictitious quantities $T^*$ and $q^*$ may be eliminated among equations (1, 2, 3). The second-order terms may be dropped, and the result is obtained that

$$M\delta q + q\delta M = q'\delta M$$

$$= -\frac{c_p}{L} \left[ M\delta T + T\delta M + MT\frac{R}{c_p} \frac{\delta p}{p} - T'\delta M \right].$$

Both sides may be divided by $M \delta p$ and the limit taken as $\delta p$ approaches zero. A convenient final form is:

$$\frac{1}{M} \frac{dM}{dp} = \frac{\frac{dq}{p} \frac{c_p}{L} \frac{dT}{dp} \frac{R}{T}}{\frac{c_p}{L} (T - T') - (q - q')}.$$  \quad (4)

Equation (4) enables the rate of mass entrainment into the cloud to be computed from any pair of soundings inside and outside the cloud.

3. The gradient of liquid water within the cloud

The flux of water substance at $p$ is $M(q + W)$. The flux of water substance entrained is $(q' + \delta q')\delta M$. The sum of these two must equal the final flux of water substance in the mixed cloud at $p - \delta p$, that is, $(M + \delta M)(q + \delta q + W + \delta W)$. If second-order terms are dropped,

$$M(\delta q + \delta W) = (q' - q - W)\delta M.$$

This may be divided by $M \delta p$; the limit as $\delta p$ approaches zero is

$$\frac{dW}{dp} = (q' - q - W) \frac{1}{M} \frac{dM}{dp} - \frac{dq}{dp}. \quad (5)$$

Equations (4) and (5) give gradients of the quantities $M$ and $W$. When the problem is to find $M$ and $W$, a numerical integration must be used, starting from some known boundary condition; for example, one may choose the pressure at condensation level $p_s$ as the place where $W(p_s) = 0$ and $M(p_s)$ is unit flux and proceed with a numerical integration from there upward. The graphical solution of this problem given in the following section illustrates the physical principles in a simple manner.
4. A graphical solution for $dM/dp$, given $q$, $T$, $q'$, $T'$

The setting of all that follows is the $q,T$-plane. Points representing the soundings inside and outside the cloud may be plotted for each pressure level $p_i$:

$$p_i = p_i - i \Delta p, \quad i = 0, 1, 2, \ldots$$

where $\Delta p$ is a finite pressure interval. Thus there are two sets of points, $(q_0, T_1)$ and $(q_1', T_1')$. For convenience smooth curves may be drawn through these points (fig. 2). Now these steps may be followed:

1. The mass of the cloud at $p_0$ is $M_0$. Imagine it moved to $p_1$ without condensation. Since $q$ remains constant, the process occurs along a vertical line. The point $(q_0, T_1''')$ is determined by the dry-adiabatic lapse rate (fig. 3).

2. Now imagine that $M_0$ at this new $T_1'''$ mixes with outside air drawn in at the level $p_1$, that is, at $(q_1', T_1')$. The temperature and humidity of the mixture must lie along the line $[(q_0, T_1''')(q_1', T_1')]$ at the point designated $(q_1^*, T_1^*)$, as may be seen in fig. 4.

3. When condensation occurs, the point representing the mixture moves along the line given by (3), the numerical value of $dT/dq$ being everywhere approximately $-2.47 \times 10^9$ C. But since this condensation process must end at the point $(q_1, T_1)$ representing the actual cloud, the line through $(q_1, T_1)$ and having the slope $dT/dq = -2.47 \times 10^9$ C determines $(q_1^*, T_1^*)$, as seen in fig. 5.

4. The point of intersection is $(q_1^*, T_1^*)$. The ratio of the lengths of the two segments equals the ratio of the flux into the cloud at this level to the upward flux at level $p_0$, that is,

$$\frac{M_1 - M_0}{M_0} = \frac{[(q_0, T_1''')(q_1^*, T_1^*)]}{[(q_1', T_1')(q_1^*, T_1^*)]}$$

The process may be continued. Perhaps the simplest technique is the following: Move $(q_1^*, T_1^*)$ down vertically to $(q_1^*, T_2'')$. Draw the line $[(q_1^*, T_2'')(q_1', T_2')]$. Finally, draw through $(q_0, T_2)$ a line having the slope $dT/dq = -2.47 \times 10^9$ C, the point of intersection being $(q_1^*, T_2^*)$, as seen in fig. 6.

5. Application to soundings near San Juan

Airplane soundings inside and outside trade-wind cumulus were obtained in the course of an expedition...
from the Woods Hole Oceanographic Institution during April 1946 under the leadership of Dr. Jeffries Wyman and Mr. A. H. Woodcock. Wet- and dry-bulb temperatures were measured; consequently values of \( q \) and \( T \) and of \( q' \) and \( T' \) can be obtained. The resulting values of \( M \) and \( W \) found by the graphical method, the pressure increment being 10 mb, are shown in table 1. The value of \( M \) at the base of the cloud is arbitrarily chosen to be unit flux. The results indicate that the amount of entrained outside air is about twice the original volume. The concentration of liquid water passes through a maximum near the middle altitude of the cloud. Observations of the concentration of liquid water, had they been made, would have provided an excellent test of the hypothesis that surrounding air is entrained into the ascending column.

### REFERENCES


